Lec 22: 11/11/2010

White Dwarfs (Cont'd):

Thermal Evolution of White Dwarfs:

Because white dwarfs have a non-zero temperature, they will have a finite luminosity due to the radiation of energy. The luminosity is determined by the nature of the physical processes that take place near the surface of the white dwarf.

The density of a white dwarf decreases from the central value ρ_c (at $r=0$) to zero at the surface ($r=R$). Lets assume the equation of state to be that of a degenerate matter for $r<r_0$, and that of an ideal gas at $r>r_0$ (r_0 being an intermediate point). At $r=r_0$ the Fermi energy of the system is equal to the thermal energy:

$$E_F = \frac{p_F^2}{2m_e} \simeq k_B T \Rightarrow \rho_0 = \frac{m_u \mu_e}{3\pi^2}\left(\frac{\hbar}{m_e c}\right)^{-3}\left(\frac{2k_B T}{m_e c^2}\right)^{3/2} \simeq 6\times10^{-9}\,\mu_e\, T_0^{3/2}$$

Here T_0 denotes the temperature at $r=r_0$. Because of the large thermal conductivity at $r<r_0$ (due to high density of free electrons), the temperature is constant for $r<r_0$. Thus $T_0 = T_{core}$.

For a radiative envelope we have:

$$\rho \propto \left(\frac{M}{L}\right)^{\frac{1}{2}} T^{\frac{13}{4}}$$

Combining the two relations in above, we find:

$$L \propto M T_0^{\frac{7}{2}} \propto M T_{core}^{\frac{7}{2}}$$

The characteristic cooling time is given by:

$$t_{cool} \simeq \frac{U}{L} \propto \frac{M T_{core}}{M T_{core}^{\frac{7}{2}}} \propto T_{core}^{-\frac{5}{2}}$$

This leads to:

$$L \propto M t_{cool}^{-\frac{7}{2}}$$

A rigorous derivation of the cooling time gives:

$$\boxed{t_{cool} \simeq 3.9 \times 10^6 \, yr \left(\frac{A}{12}\right)^{-1} \left(\frac{M}{M_\odot}\right)^{\frac{5}{7}} \left(\frac{L}{L_\odot}\right)^{-\frac{5}{7}}}$$

- If $N(L)$ denotes the number density of white dwarfs in the range of luminosities $(L, L+dL)$, the rate of change of this number density follows;

$$\dot{N} = \frac{dN}{dL}\dot{L} = \left(\frac{\dot{L}}{L}\right)\left(\frac{dN}{d\ln L}\right) = \frac{1}{t_{Cool}}\left(\frac{dN}{d\ln L}\right) \propto \left(\frac{L}{M}\right)^{\frac{5}{7}}\left(\frac{dN}{d\ln L}\right)$$

If the white dwarfs are produced at an approximately constant rate, we expect $\dot{N} \approx$ Const., which results in;

$$\frac{dN}{d\ln L} \propto \left(\frac{L}{M}\right)^{-\frac{5}{7}}$$

The coolest white dwarfs with $L \simeq 10^{-4.5} L_\odot$ have a cooling time of $\sim 10^{10}$ yr. The observed luminosity function for the white dwarfs rapidly falls for $L \lesssim 10^{-4.5} L_\odot$. This is consistent with an age of $(9 \pm 1.8) \times 10^{9}$ yr for the galaxy. Adding the time spent in the pre-white dwarf stage ~~in~~ the stellar evolution would suggest that star formation in the disk of the galaxy began

$(9.3 \pm 2) \times 10^9$ yr ago. This is $\sim 6 \times 10^9$ yr shorter than the age of the globular clusters, and has implications for the formation scenarios of the disk and globular clusters.

One comment is in order at this point. The electronic contribution to the thermal energy becomes important at lower temperatures, and hence must be considered during the late stages of cooling. This can be understood as follows.

At any given temperature T, a fraction $\frac{k_B T}{E_F}$ of the electrons can be thermally excited (those that are sufficiently close to the Fermi surface). Each of these electrons carry an energy $\sim k_B T$, and hence the thermal energy per electron is $\frac{(k_B T)^2}{E_F}$.

On the other hand, at high temperatures the thermal energy per ion is $k_B T$. As the plasma cools, the ions crystallize and form a lattice. For a lattice, the ions move collectively that can be thought of as vibrations. These have a characteristic Debye temperature Θ, which follows:

$$k_B \Theta = \hbar \omega_p \quad , \quad \omega_p = \frac{2e}{m_u}\left(\frac{\pi Z \rho}{A}\right)^{\frac{1}{2}} \quad (\omega_p: \text{plasma frequency})$$

The thermal energy per ion at $T \lesssim \Theta$ is $k_B T \left(\frac{T}{\Theta}\right)^3$, with $\left(\frac{T}{\Theta}\right)^3$ factor due to the fact that the specific heat is $\propto T^3$ at low temperatures.

It is therefore seen that at high temperatures electronic contribution to the thermal energy is negligible compared to that from ions $\left(\frac{k_B^2 T^2}{E_F} \text{ vs } k_B T\right)$. While, at sufficiently low temperature it will dominate $\left(\frac{k_B^2 T^2}{E_F} \text{ vs } \frac{k_B T^4}{\Theta^3}\right)$. For a $1\,M_\odot$ Carbon white dwarf, $\Theta \simeq 2\times10^7$ K, and the electronic contribution dominates at $T \lesssim 2\times10^6$ K. These effects are to be taken into account by numerical calculations.

Equation of State at Higher Densities:

Stars with an initial mass $M \gtrsim 8 M_\odot$ have a core made of heavier elements that $^{12}C, ^{16}O$ (such as iron) with core densities $\rho \gtrsim 10^{13}\ g\ cm^{-3}$ and temperatures $T \gtrsim 10^{10}$ K. The resultant structure is determined by the equation of state for matter at high densities.

We consider a system that has a mixture of nuclei, free electrons and free neutrons. The energy density of such a system is:

$$\epsilon = n_N M(A,Z) + \epsilon'_e(n_e) + \epsilon_n(n_n)$$

Here the subscripts N, e, n refer to nuclei, electrons, and neutrons respectively. Also:

$$M(A,Z) = \left[(A-Z)\, m_n c^2 + Z(m_p + m_e)\, c^2 - A E_b\right]$$

E_b is the binding energy (per nucleon) of a nucleus. Note that ϵ'_e includes kinetic energy of free electrons alone since their rest mass energy is already included in $M(A,Z)$. There is an empirical relation for $M(A,Z)$:

$$\boxed{\begin{gathered} M(A,Z) = m_u c^2 \left[b_1 A + b_2 A^{\frac{2}{3}} - b_3 Z + b_4 A \left(\frac{1}{2} - \frac{Z}{A}\right)^2 + \frac{b_5 Z^2}{A^{\frac{1}{3}}} \right] \\ b_1 = 0.991749 ,\ b_2 = 0.01911 ,\ b_3 = 0.000840 ,\ b_4 = 0.10175 ,\ b_5 = 0.000763 \end{gathered}}$$

Defining the baryon number density $n = n_N A + n_n$ and the fraction of free neutrons as $Y_n \equiv \frac{n_n}{n}$, we have:

$$\varepsilon = n(1 - Y_n) \frac{M(A,Z)}{A} + \varepsilon_e'(n_e) + \varepsilon_n(n_n)$$

$$n_e = n(1 - Y_n) \frac{Z}{A}$$

$$n_n = n Y_n$$

Now, we have to minimize ε with respect to A, Z, Y_n. This is done by setting partial derivatives of ε with respect to these variables to zero. We then find the following equations:

$$\begin{cases} b_3 + b_4\left(1 - \frac{2Z}{A}\right) - 2b_5 \frac{Z}{A^{\frac{1}{3}}} = \left[(1+x_e^2)^{\frac{1}{2}} - 1\right]\frac{m_e}{m_U} & * \\ Z = \left(\frac{b_2}{2b_5}\right)^{\frac{1}{2}} A^{\frac{1}{2}} & ** \\ b_1 + \frac{2 b_2 A^{-\frac{1}{3}}}{3} + b_4\left(\frac{1}{4} - \frac{Z^2}{A^2}\right) - \frac{b_5 Z^2}{3A^{\frac{4}{3}}} = (1+x_n^2)^{\frac{1}{2}}\frac{m_n}{m_U} & *** \end{cases}$$

Where $x_e \equiv \frac{P_{F,e}}{m_e c}$ and $x_n = \frac{P_{F,n}}{m_n c}$.

For a given value of A (>56) we can find Z from equation **. Knowing Z, A we can then find x_e from equation *. Equation *** can be used to determine x_n. If $x_n^2 > 0$, free neutrons exist. Otherwise there will be no free neutrons.

At this point we have the total energy density, and the total pressure (from nuclei, electrons, neutrons). This will give us the equation of state of the system.

A brief summary of physical processes that occur at high densities can be given as follows:

- At $\rho \gtrsim 10^6$ g cm^{-3} the degenerate electron gas becomes relativistic. Deviations from an ideal degenerate gas arise for $\rho \gtrsim 10^7$ g cm^{-3} as discussed before.
- At $\rho \gtrsim 10^{10}$ g cm^{-3} neutronization occurs due to electron capture by protons inside nuclei. This moves the nuclei to the neutron rich side.
- At $\rho_{drip} \sim 10^{12}$ g cm^{-3} neutron drip occurs. This corresponds to neutron rich nuclei with $A = 122$ and $Z = 39$. The Fermi energy of electrons is ≈ 23.6 MeV at this density. Free neutrons contribute more and more above this density. For $\rho \gtrsim 4 \times 10^{12}$ g cm^{-3}, they provide most of the pressure, and the equation of state can be approximated by that of a cold gas of electrons, protons, and neutrons that are in equilibrium (through β and inverse β decay).

– At $\rho_{nucl} \simeq 2.8 \times 10^{14}\ g\ cm^{-3}$ the individual nuclei begin to interact strongly and dissolve together. The equation of state above these densities is uncertain and reliable conclusions are difficult to arrive at. Some of the possibilities will be discussed later on.